1st Junior Penchick Online Mathematical Olympiad

Final Stage, Senior Division

(Ages 18 and below)

4 July 2026

INSTRUCTIONS AND INFORMATION

1. **There are 17 questions to be answered in 120 minutes, split into two parts.** The highest possible score for this contest is 65 points.
   - *Part I (short answer):* Submit the best answer to each question. Three points are given for a correct answer, for a total of 45 points for this section.

     For this paper, all answers are **integers** from 0 to 999 inclusive.
   - *Part II (problem-solving):* The last 2 questions require you to show full solutions in order to get full credit. You can gain a maximum of 10 points for each question depending on the accuracy and completeness of your submission, for a total of 20 points for this section.
2. **Always exhibit academic honesty during the contest.** Do not use calculators, protractors, graph paper, search engines, artificial intelligence, and other external applications, resources, or help while answering the contest. Rulers, compasses, and blank or lined paper are allowed. Cheating in any form is strictly prohibited and will be acted upon based on the violation system.

Enjoy the contest!

Part I: Short-Answer Questions

Submit the best answer to each question. Three points are given for a correct answer, for a total of 45 points for this section.
For this paper, all answers are **integers** from 0 to 999 inclusive.

1. If the value of $\sqrt[3]{44+18\sqrt{6}}-\sqrt[3]{44-18\sqrt{6}}$ can be simplified into $a\sqrt{b}$ for positive integers a,b, where b does not have any perfect square divisors other than 1, find the value of $a+b$.

2. A T-shirt reads "I ♡ penchicks," where the heart symbol is formed by an equilateral triangle of side length 6 cm together with two semicircles, each having a diameter equal to a side of the triangle, attached to the two upper sides of the triangle to form the rounded top of the heart. If the area of the heart can be written in the form

$$a\sqrt{b}+c\pi$$

for integers a,b,c, where b does not have any perfect square divisors other than 1, find the value of $100a+10b+c$.

3. Call a positive integer n a *bizarre* number if n satisfies $n^3 \equiv n \pmod{30}$. If the number of bizarre positive integers less than 2026 is B, find the last three digits of B.

4. Two robots, Travistron and Phoebot, each independently pick a random positive integer from 1 to 127 (in base ten), inclusive, which they then write in binary. The probability that the binary representation of Travistron's number has more digits than that of Phoebot's number is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. Let a, b, and c be real numbers such that

$$a-2b+c=4 \quad \text{and} \quad 3a-b-c=6.$$

Then find the value of $a^2+b^2-c^2$.

6. Points A,B,C,D lie on a circle in clockwise order such that $AB \parallel CD$. The diagonals AC and BD intersect at point E. A line through E parallel to AB meets AD at F and BC at G. If $AB=24$ and $CD=8$, find the length of FG.

7. Let a and b be positive integers, and let

$$n=\gcd(a,b)+\operatorname{lcm}(a,b)+a+b.$$

Determine the last 3 digits of the sum of all possible distinct values of n less than 2026.

8. Find the minimum value of

$$f(x)=16\tan x+\frac{27}{\tan^2 x}, \qquad \text{where } 0<x<\frac{\pi}{2}.$$

9. One day, Jobert received an update: the *Cyberpunk: Edgerunners* $\times$ *Wuthering Waves* gacha collab is live! Here is how 5-star pulls work during the event:

- Each 5-star pull gives you a 50% chance of winning the featured character, Lucy, straight from the Cyberpunk collab.
- If you lose that 50/50, the game picks uniformly from the standard 5-star pool, but *Wuthering Waves* secretly slips Lucy back into that pool too, so even on a "loss" you still have a $\frac{1}{6}$ chance of landing on Lucy.
- If the second stage also fails, you are guaranteed to pull Lucy on your very next 5-star pull.

Suppose a player pulls two 5-star characters back-to-back without any 5-star pulling history, and the second one turns out to be Lucy. Let the probability that the first pull was also Lucy be the fraction $\dfrac{m}{n}$, where m and n are relatively prime integers. Find $m+n$.

10. Let $\triangle ABC$ be a triangle where $AB = 16$, $BC = 12$, and $AC = 20$. Let the perpendicular bisector of $\overline{AC}$ intersect $\overline{AB}$ at D and the circumcircle of $\triangle ABC$ at E and F. Find the value of $|DF - DE|$.

11. Renren wants to cross a grid of glaciers with lattice points from $(0,0)$ to $(4,4)$, starting at the bottom-left corner and ending at the top-right corner. He can only move either up or to the right at each step. However, Penrick has broken 1 glacier in the grid, at a lattice point that is neither Renren's starting nor ending point. Assuming Penrick chooses the placement of the broken glacier to minimize the number of paths, what is the least number of ways that Renren can reach the top-right corner?

12. Let $P(x)$ be a Penrick polynomial with integer coefficients satisfying $P(1) = 53$ and $P(5) = 69$. Given that $P(13)$ can take various integer values depending on the choice of P, determine the least positive possible value of $P(13)$.

13. Pelican was writing down N, a four-digit number in base b, written as $(2431)_b$ (where necessarily $b \geq 5$, since 4 must be a valid digit). When N is converted to base 10, it is divisible by the sum of its digits in base b. Pelican calls these bases b as *birdie bases*. Find the 3 leftmost digits of the sum of all integer bases $b < 2026$ that are birdies.

14. In triangle ABC, point D lies on $\overline{BC}$ with $BD = 2$ and $DC = 3$. Point E lies on $\overline{AC}$ with $AE = 1$ and $EC = 4$. Cevians $\overline{AD}$ and $\overline{BE}$ intersect at point P. Line CP, extended, meets $\overline{AB}$ at point F. If the area of triangle ABC is S, the area of triangle APF equals $\dfrac{m}{n} \cdot S$ for positive integers m, n with $\gcd(m,n) = 1$. Find $m+n$.

15. Find the last three digits of the smallest integer $x \geq 2026$ such that

$$\frac{x^2+x+1}{x^{38}-x^{37}+10}$$

is not already in lowest terms (i.e., the numerator and denominator share a common factor greater than 1).

Part II: Proof Questions

The last 2 questions require you to show full solutions in order to get full credit. You can gain a maximum of 10 points for each question depending on the accuracy and completeness of your submission, for a total of 20 points for this section.

1. Find all pairs of positive integers (a,b), such that $a!+b$ and $3((a+1)!-a!-a^2+b)$ are both divisible by $a+b$.

   To get full credit for this question, you must give the correct answer, and do the following:

   - *Show that all possible pairs you provided satisfy the conditions.*
   - *Show a full proof why there are no other possible pairs.*

2. Chubby and Delta encounter the PenchickMagician, who lays $2n$ cards in a row, where $n > 1$ is an odd integer. Exactly n of the cards are Copper and the remaining n are Diamond.

   The PenchickMagician shuffles the cards into an arbitrary arrangement. Chubby then challenges Delta to transform the cards into any arrangement Delta desires using only the following operation:

   Flip: Choose any contiguous block of n cards and reverse its order.

   Determine all odd integers $n > 1$ for which Delta will always be able to reorder the cards to any arrangement using this operation.

   For example, when $n = 5$,

   $$\text{CDCDD}\,\underbrace{(\text{CCDCD})}_{\text{chosen block}} \longrightarrow \text{CDCDD}\,\underbrace{(\text{DCDCC})}_{\text{after flipping}},$$

   since the chosen block is reversed.

   To get full credit for this question, you must give the correct answer, and do the following:

   - *For all the possible values of n, provide a winning strategy that Delta can use.*
   - *For the others values of n, show a full proof on why Delta cannot guarantee a win.*

Answer Key

Part I: Short-Answer Questions

1. 8

2. 939

3. 215

4. 169

5. 4

6. 12

7. 154

8. 36

9. 158

10. 15

11. 30

12. 5

13. 720

14. 149

15. 116