1st Junior Penchick Online Mathematical Olympiad

Final Stage, Sample Problems

4 July 2026

INSTRUCTIONS AND INFORMATION

Each division has **six sample problems** that you can use to prepare for the JPOMO. The answers and solutions to each item can be found starting on page 4.

The sample problems (and the problems in the actual test) are of two types:

- *Short-answer:* Submit the best answer to each question.

  For the Primary division, answers are either **whole numbers** or **fractions** (in the form of common fractions, mixed fractions, or decimals). All answers must be exact and in their simplest form; otherwise, they will be marked as incorrect. For example, if the answer is $\frac{5}{3}$, then the answers $\frac{5}{3}$, $1\frac{2}{3}$, and $1.\overline{6}$ are accepted (for the decimal, the answer is accepted as long as it is indicated that the decimal is repeating, for example $1.66666\ldots$). However, the answer $\frac{10}{6}$ is not accepted because it is not simplified, and the answer 1.67 is not accepted because it is an estimation of the answer.

  For the Junior and Senior divisions, all answers are **integers** from 0 to 999 inclusive.

- *Proof:* These questions require you to show full solutions in order to get full credit. Solutions in the actual test are graded depending on the accuracy and completeness of your submission.

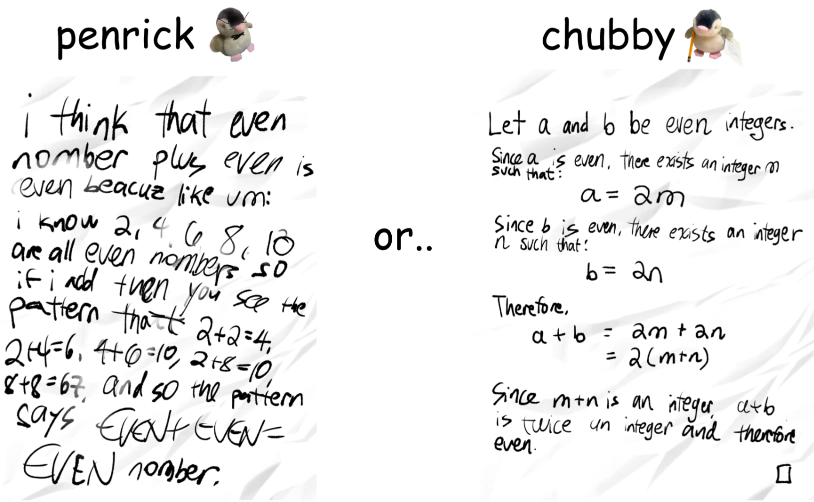
Enjoy the problems!

Contents

1 Introduction to Solution Writing

In the JPOMO Final Stage, you will be writing full solutions for two of the questions. This is a list of important tips that you can use when writing your solutions. We will also be going over examples of how to write your solutions during the lectures that will take place a few days before the contest.

Before proceeding, consider the following scenario. You are given two proofs, shown below, of the statement that the sum of two even numbers is an even number. Which one would you prefer to read, and why?



1.1 Be Crystal Clear

It is important to remember while you are writing that **someone else is going to read your work**. Since you are writing solutions as a submission for a contest, it is usually fine to write **just enough so that the grader can understand your work** (the case is different if you will write for a larger audience).

Ideally, you should be writing solutions **legibly** and **in sentences**. This way, you can be mostly confident that the grader will understand your solution. However, if you are running out of time, it is okay to not write a few parts (especially calculations) in sentences, as long as what you are writing is still clear.

If you use terms that only you understand (or that only make sense in your head) without explaining what they mean, then the grader will likely not understand your work and you will not get as much points as you hoped. Therefore, **define any new variables or terms that you will use in your solution**.

1.2 Explain Your Reasoning

If you are asked to **prove** or **show** a certain statement, then you must fully explain why this statement is true. It is **not enough** to provide a few examples that support what you are trying to prove. For example, if the problem is:

Prove that for any positive integer n, the sum of the first n odd numbers is n^2.

then do not do this:

> *We show a few examples to see that the sum is indeed n^2:*
>
> $$\begin{aligned} 1 &= 1 = 1^2, \\ 1+3 &= 4 = 2^2, \\ 1+3+5 &= 9 = 3^2. \end{aligned}$$
>
> *Therefore, for any positive integer n, the sum of the first n odd numbers is n^2.*

This "proof" does not work because you only proved the statement for three values of n and not *all* of them: what if the sum of the first four odd integers is not 4^2?

However, if the question asks you to prove that something **exists**, then it would be fine to provide examples, **as long as you show that the examples satisfy the given conditions**. For example, if the problem is:

> *Prove that there is a natural number that is divisible by both 2 and 3.*

then the following would already be a complete proof:

> *We claim that the number 12 is a natural number that is divisible by both 2 and 3.*
>
> *Since $12 \div 2 = 6$ and $12 \div 3 = 4$ are both integers, we conclude that 12 is indeed divisible by both 2 and 3.*

Of course, proofs for actual problems will likely be much longer than this.

1.3 Write Correct Math

It is always important to **write correct math**; one small mistake can affect the rest of your solution! For example, in Penrick's proof on the previous page, he wrote $8+8=67$, which is not true, and writing this does not support the claim at all (since 67 is odd). Thus, if you still have remaining time in a contest, it is always worth it to **check that everything you wrote in your solution is correct** and that you made no small errors like typos.

Relating to this is the following:

1.4 Write in the Correct Direction

A common mistake that students often do is **writing proofs backwards**: this means that the proof goes from what is being asked to something that is known to be true. This is incorrect because "if A then B" does not always mean "if B then A". (These statements are called **converses** of each other.) For example, a shape being a square means that it is also a rectangle, but a shape being a rectangle does not always mean that it is a square.

Consider this classic problem:

> *Prove that for positive numbers a and b, we have*
>
> $$\frac{a+b}{2} \geq \sqrt{ab}.$$

Do not present your proof like this:

> *Multiplying both sides by 2 gives $a+b \geq 2\sqrt{ab}$, and squaring both sides gives*
>
> $$a^2 + 2ab + b^2 \geq 4ab.$$
>
> *Subtracting $4ab$ from both sides gives*
>
> $$a^2 - 2ab + b^2 \geq 0.$$
>
> *Since the left side is equal to $(a-b)^2$, we have $(a-b)^2 \geq 0$, which we know is true because the square of any number is nonnegative.*

Instead, write it in the other direction:

> *Since the square of any number is nonnegative, we have $(a-b)^2 \geq 0$, and expanding the left side gives*
>
> $$a^2 - 2ab + b^2 \geq 0.$$
>
> *Adding $4ab$ to both sides gives*
>
> $$a^2 + 2ab + b^2 \geq 4ab,$$
>
> *and factoring the left side gives $(a+b)^2 \geq 4ab$. Finally, since a and b are positive, we take the square root of both sides to get $a+b \geq 2\sqrt{ab}$, and divide both sides by 2 to get*
>
> $$\frac{a+b}{2} \geq \sqrt{ab}.$$

You may notice that the steps we used here are **reversible**, meaning we can do the steps in both directions without any issues. However, there are many times when the steps are not reversible: for example, we can multiply both sides of an equation by x but not divide both sides by x (since we do not know if x is nonzero).

1.5 Some Other Types of Problems

Problems of some types have specific components that would be expected of your solution. We already discussed an example of this, which is when the problem asks to show that something exists. Here are two other common types of problems:

- If the problem asks you to **find all** objects that satisfy a certain property, you must:
  1. state what these objects are,
  2. show that they satisfy the property, and
  3. show that there are no other objects that satisfy the property.
- If the problem asks you to find a **minimum or maximum value** of a certain quantity y, you must:
  1. state what the minimum or maximum value is,
  2. show that the value is attainable (the value M can be achieved by assigning $x = A$, which leads to $y = \ldots$), and

3. show that there cannot be a value lower than (if you are asked for a minimum) or higher than (if you are asked for a maximum) your answer.

If you forget these pointers, do not worry; the proof questions in the JPOMO Final Stage will include these as a guide.

For more comprehensive tips, you may check out `https://web.evanchen.cc/handouts/english/english.pdf`, although the examples shown there are more advanced.

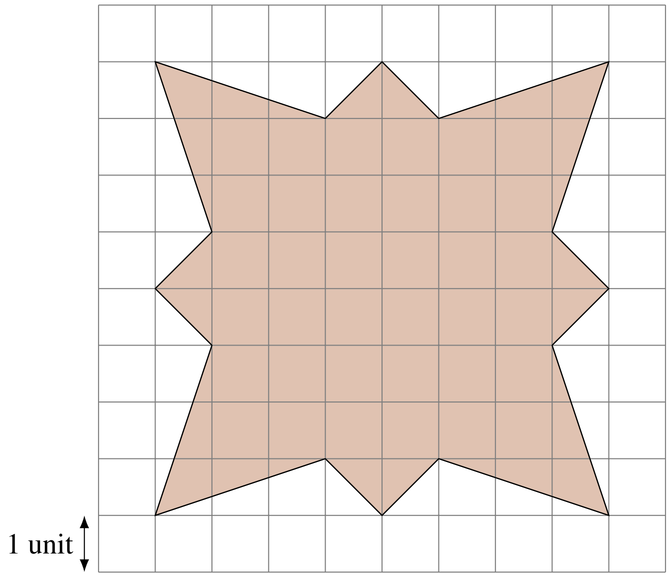
Enjoy solution writing!

2 Primary Division

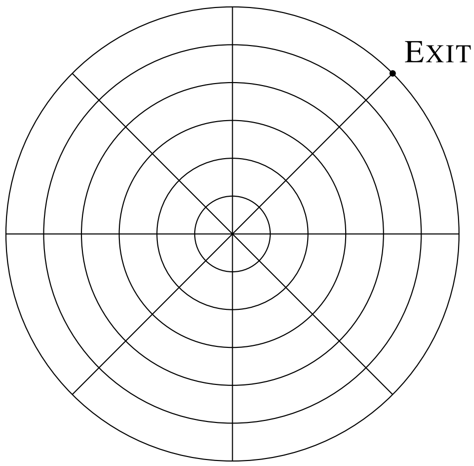
2.1 Short-Answer Questions

1. *(1st JPOMO Qualifying Stage, Paper A)* Chubby, working hard, studies for $\frac{5}{9}$ hours today and adds $\frac{1}{9}$ hours each day onward. Pelican, hardly working, studies $\frac{4}{9}$ hours today and cuts $\frac{1}{9}$ hours each day instead. Solve for the total number of hours they studied together today and over the next three days.

2. *(1st JPOMO Qualifying Stage, Paper B)* Given the side length of each square on the grid below is 1 unit, find the area of the shaded region.



3. *(1st JPOMO Qualifying Stage, Paper B)* Oh no! Penrick has trapped Delta in the maze shown below! Delta is currently at the center of the six circles and must go to the point marked EXIT to exit the maze. She likes to explore, so she can possibly take detours by going around any of the circles. If Delta can only move in the gridlines, outward (away from the center) or around any of the circles without revisiting any point, then she can exit the maze in N ways. How many prime factors does N have?



4. *(1st JPOMO Qualifying Stage, Paper B)* Let A, B, D, E, H, R, Y be distinct nonzero digits such that

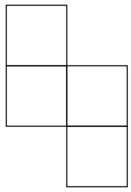
$$\overline{BREAD} + \overline{HEAD} = \overline{READY},$$

where $\overline{BREAD}$, $\overline{HEAD}$, and $\overline{READY}$ have five, four, and five digits, respectively. It is guaranteed that only one solution exists. Calculate

$$B+R+E+A+D.$$

2.2 Proof Questions

5. Call a tile a *squiggly tile* if it has the following shape.



Find all possible pairs of side lengths greater than 2 such that a rectangle with those side lengths can be tiled completely only using *squiggly tiles*, if:

(a) Squiggly tiles can only be rotated and moved.

(b) Squiggly tiles can also be reflected/flipped.

6. There are 15 coins arranged in a row, all pointing up heads. One can flip exactly four coins that are all grouped up and adjacent to each other, changing what they point up (either from heads to tails or from tails to heads). Find a way to flip all coins such that they all point up tails, or show that it cannot be achieved.

3 Junior Division

3.1 Short-Answer Questions

1. Find the number of positive integers x from 3 to 999, inclusive, such that

$$12_x \cdot 12_{x+1} = 110_{x+2}.$$

(The notation A_b means that the number A is expressed in base b.)

2. *(1st JPOMO Qualifying Stage, Paper D)* Let α, β, γ be the roots of the cubic equation

$$x^3 - 7x^2 + 3x + 2 = 0.$$

If the cubic equation $x^3 + px^2 + qx + r = 0$, for some integers p, q, and r, has roots

$$\alpha + \beta, \quad \alpha + \gamma, \quad \beta + \gamma,$$

find $p + q + r$.

3. *(1st JPOMO Qualifying Stage, Paper D)* The 12 Chrysos Heirs of Amphoreus are seated around a round table. All heirs are distinct. Due to internal conflicts, the following conditions must be satisfied:

- Anaxa and Aglaea must sit directly opposite each other.
- Hyacine must sit adjacent to Aglaea.
- Phainon and Mydei must not sit next to each other.

Let N be the total number of distinct seating arrangements that satisfy these conditions. Find the remainder when N is divided by 1000.

4. *(1st JPOMO Qualifying Stage, Paper C)* In trapezoid $ABCD$ with $AB \parallel CD$, BD and AC are diagonals with AC being perpendicular to both AB and CD. If $\angle B = 45^\circ$ and $\angle D = 30^\circ$, then $\left(\frac{BD}{AC}\right)^2$ can be written in the form $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by any perfect square. What is $100c + 10b + a$?

3.2 Proof Questions

5. Let ABC be a triangle and let D be an arbitrary point on the perpendicular bisector of $\overline{BC}$. Suppose the internal angle bisectors of $\angle ADB$ and $\angle ADC$ intersect $\overline{AB}$ and $\overline{AC}$ at X and Y, respectively. Show that $\overline{XY}$ is parallel to $\overline{BC}$.

6. *(1st JPOMO Qualifying Stage, Paper C, modified)* Pelican, Chubby and Renren each list an arithmetic sequence. Pelican's sequence begins with the following:

$$4, 11, 18, \ldots$$

Chubby's sequence begins with the following:

$$13, 28, 43, \ldots$$

and Renren's sequence begins with the following:

$$6, 25, 44, \ldots$$

Determine, with proof, all positive integers that appear in all three sequences.

4 Senior Division

4.1 Short-Answer Questions

1. A kinetic sculpture rests on three adjustable supports of positive lengths x, y, and z. For the sculpture to remain balanced, the supports must satisfy $2x+y+z=12$. The sculpture's stability score is given by x^2yz. Find the greatest possible value of the stability score.

2. Let $ABCD$ be a cyclic quadrilateral whose diagonals intersect at E. Suppose $AB=9$, $CD=9$, $DA=16$, and AC bisects $\angle BAD$. Find $AC+BD$.

3. *(1st JPOMO Qualifying Stage, Paper E, updated)* Let a and b be positive integers such that
$$\gcd(a,b)=12.$$
Suppose that
$$\gcd(3a+2b,\ 5a+7b)=132.$$
What is the smallest possible value of $a+b$?

4. *(1st JPOMO Qualifying Stage, Paper E)* A dart is thrown uniformly at random at a circular target of radius 100. The target is divided into 100 rings by circles of radii $1,2,3,\ldots,100$, each one fully in all circles which are larger. Starting from the center, the rings are alternately assigned a score of $+1$ and -1, with the innermost disk assigned $+1$. The expected value of the score of a random throw can be expressed as $\frac{m}{n}$, where m and n are relatively prime integers and n is positive. Find $m+n$.

4.2 Proof Questions

5. Let $n!=1\cdot 2\cdot 3\cdots n$.
   (a) Prove or disprove: For every integer $n>1$, the rightmost nonzero digit of $n!$ must be even.
   (b) Determine all digits $x\in\{1,2,3,4,5,6,7,8,9\}$ such that there exist infinitely many integers n for which the rightmost nonzero digit of $n!$ is equal to x.

6. A function determines the number of penguins standing in a fishmonger after t hours. The number of penguins who have entered the shop is given by
$$\left\lfloor \frac{t^2}{2} \right\rfloor,$$
and the number of penguins who have left after t hours is
$$\left\lfloor \frac{t^2}{5} \right\rfloor.$$
The shop can fit at most 30 penguins at a time. After how many minutes is the number of penguins inside greater than the shop's capacity?

   The floor function $\lfloor x \rfloor$ rounds x down to the nearest integer less than or equal to x. For example,
$$\lfloor 2.718 \rfloor = 2, \quad \lfloor \pi \rfloor = 3, \quad \lfloor -4.1 \rfloor = -5.$$

5 Answer Key

Answers and solutions for the Senior division will be released soon.

Primary Division

1. 4
2. 48
3. 2
4. 28

Junior Division

1. 997
2. 15
3. 640
4. 325

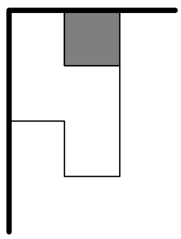
6 Solutions

Solutions for the short-answer questions will be released soon.

6.1 Primary Division

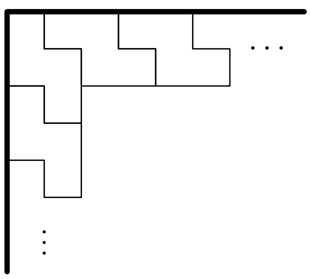
5. (a) We claim that it is $\boxed{\text{impossible}}$ to completely tile any rectangle with squiggly tiles if they can only be rotated and moved.

Suppose we place one squiggly tile to cover one corner of the rectangle, as shown below. Then, there is no way to place a squiggly tile to cover the shaded region. Therefore, the rectangle cannot be completely covered by squiggly tiles.

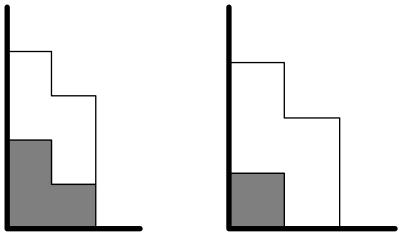


(b) We claim that it is once again $\boxed{\text{impossible}}$ to completely tile any rectangle with squiggly tiles if they can be rotated, moved, and reflected.

Suppose we place just enough squiggly tiles in the configuration shown below to cover as much area as possible.



Then, there would be a region (shaded in gray) below or to the right of the squiggly tiles farthest from the top-left corner that cannot be covered by a squiggly tile. This region takes the form of any of the two regions shown below. Therefore, the rectangle cannot be completely covered by squiggly tiles.



6. We claim that we $\boxed{\text{cannot}}$ achieve an arrangement where all coins point up tails. We use the following **lemma** (something that helps us prove the statement):

   Lemma: The number of coins that show heads must always change by an even amount.

   We split the proof of this lemma into five cases, considering all possible flips and showing they all follow our lemma, where our cases depend on how many coins that are selected for a flip initially showed up heads. (In the table below, a negative change means that there are fewer heads after the flip.)

No. of initial heads	No. of new heads	No. of new tails	Change in no. of heads
0	4	0	$4-0=0$
1	3	1	$3-1=2$
2	2	2	$2-2=0$
3	1	3	$1-3=-2$
4	0	4	$0-4=-4$

   Since all of the numbers in the last column are even, we conclude that the number of coins that show heads must always change by an even amount. $\square$

   Since 15 is odd, we know that there must always be an odd number of heads at any time. Therefore, we cannot have zero heads, making it impossible to make all coins point up tails.

   Note: The word "flip" in this solution is referring to the four coins selected being flipped.

6.2 Junior Division

5. By the angle bisector theorem on triangles DAB and DAC, with angle bisectors $\overline{DX}$ and $\overline{DY}$, respectively, we have $DA/AX = DB/BX$ and $DA/AY = DC/CY$. These are equivalent to $DB/DA = BX/AX$ and $DC/DA = CY/AY$, respectively. Note that since D is on the perpendicular bisector of $\overline{BC}$, we have $DB = DC$, so

$$\frac{BX}{AX} = \frac{DB}{DA} = \frac{DC}{DA} = \frac{CY}{AY}.$$

   Since $AB = AX + BX$ and $AC = AY + CY$, we have

$$\frac{AB}{AX} = \frac{AX+BX}{AX} = 1 + \frac{BX}{AX} = 1 + \frac{CY}{AY} = \frac{AY+CY}{AY} = \frac{AC}{AY}.$$

   Since $AB/AX = AC/AY$ and $\angle BAC = \angle XAY$, we have $\triangle BAC \sim \triangle XAY$, so $\angle ABC = \angle AXY$. We therefore conclude that $\overline{XY} \parallel \overline{BC}$.

6. The only positive integers that appear in all three sequences are those that can be expressed in the form $\boxed{1995a + 823}$ for any $a \in \mathbb{N} \cup \{0\}$.

The sequences are defined by

$$\begin{aligned} a_n &= 7n-3 = 7(n-1)+4, \\ b_n &= 15n-2 = 15(n-1)+13, \\ c_n &= 19n-13 = 19(n-1)+6, \end{aligned}$$

respectively. So, for a positive integer N to appear in all sequences, it must satisfy the following:

$$\begin{aligned} N &\equiv 4 \pmod{7}, \\ N &\equiv 13 \pmod{15}, \\ N &\equiv 6 \pmod{19}. \end{aligned}$$

Since $1995 = 7 \cdot 15 \cdot 19$ and 823 indeed has remainders of 4, 13, and 6 when divided by 7, 15, and 19, respectively, we know that all positive integers of the form $1995a + 823$ appear in all sequences.

Now we show that these are the only such numbers. From the first congruence, N can be expressed as $7k+4$ for some $k \in \mathbb{N} \cup \{0\}$, and substituting into the second congruence gives

$$7k+4 \equiv 13 \pmod{15} \implies 7k \equiv 9 \pmod{15} \implies k \equiv 12 \pmod{15}$$

Thus, k can be expressed as $k = 15\ell + 12$ for some $\ell \in \mathbb{N} \cup \{0\}$, so

$$N = 7(15\ell+12)+4 = 105\ell + 88.$$

We then have from the third congruence that

$$105\ell + 88 \equiv 6 \pmod{19} \implies 10\ell \equiv 13 \pmod{19} \implies \ell \equiv 7 \pmod{19},$$

so $\ell = 19m+7$ for some $m \in \mathbb{N} \cup \{0\}$. Therefore:

$$N = 105(19m+7)+88 = 1995m + 823.$$